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BRST Quantization of the Siegel Action

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1. INTRODUCTION

Self-dual fields in one-space, one-time dimension called chiral bosons[(1–10)] are basic ingredients of some string theories.[(1)] They also play an important role in studies of the quantum Hall effect[(2)] and have attracted wide interest recently.[(1–10)] These fields satisfy the self-duality condition $\dot{\phi} = \phi'$, where the overdot and prime denote time and space derivatives, respectively. Siegel[(5)] proposed an action of a two-dimensional, doubly self-dual, chiral boson.[(5–10)] Modifications of the Siegel action achieved by the addition of appropriate Wess–Zumino terms to the action have been considered in the literature.[(8,9)] The Becchi–Rouet–Stora and Tyutin (BRST) quantization of the Siegel action modified by the inclusion of an extra "Liouville term" to the original action has been studied in ref. 9. In ref. 10 we studied the Hamiltonian formulation[(11)] of the Siegel action in various gauges. In the present work we study the BRST quantization[(12,13)] of this theory[(5)] (without any modifications) following ref. 13. After briefly recapitulating the basics of the model in Section 2, its BRST formulation is investigated in Section 3.

2. BASICS OF THE MODEL

Siegel's (second-order) action in one-space, one-time dimension is[(5,6,10)]

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$$S = \int \mathscr{L}\, dt\, dx \tag{2.1a}$$

$$\mathscr{L} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\phi'^2 + \lambda(\dot{\phi} - \phi')^2 \tag{2.1b}$$

We use the Lorentz metric $g^{\mu\nu} := \text{diag}(+1, -1)$, and overdots and primes denote time and space derivatives, respectively. As a consequence of the Euler–Lagrange equations for (2.1) one has

$$\dot{\phi} = \phi' \qquad \text{and} \qquad \ddot{\phi} = \phi'' \tag{2.2}$$

The dynamics of the chiral boson is contained in (2.2). The theory is seen to possess one primary constraint:

$$\Omega_1 := p_\lambda \approx 0 \tag{2.3}$$

The total Hamiltonian density corresponding to (2.1) is[10]

$$\mathscr{H}_T = \frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2 - \frac{\lambda}{1 + 2\lambda}(\pi - \phi')^2 + p_\lambda u \tag{2.4}$$

where u is a Lagrange multiplier field. The time evolution of Ω_1 leads to a secondary constraint

$$\tilde{\Omega}_2 = \frac{1}{(1 + 2\lambda)^2}(\pi - \phi')^2 \approx 0 \tag{2.5}$$

which is classically equivalent to[10]

$$\Omega_2 = (\pi - \phi') \approx 0 \tag{2.6}$$

The matrix of the Poisson brackets of Ω_1 and Ω_2 (using the conventions of ref. 10) is

$$R_{\alpha\beta} := \{\Omega_\alpha, \Omega_\beta\}_P = \begin{bmatrix} 0 & 0 \\ 0 & -2\delta'(x-y) \end{bmatrix} \tag{2.7}$$

Here, the matrix $R_{\alpha\beta}$ is clearly singular, implying that the constraints Ω_1 and Ω_2 *together* form a set of first-class constraints. The first-class nature of the set of onstraints Ω_1 and Ω_2, in turn, ensures the gauge invariance of the theory [S(2.1)] which is seen to be invariant under the so-called Siegel gauge transformations[5,6,10]

$$\delta\phi = \epsilon(\dot{\phi} - \phi') \tag{2.8a}$$

$$\delta\lambda = -\tfrac{1}{2}(\dot{\epsilon} + \epsilon') + \epsilon(\dot{\lambda} - \lambda') - \lambda(\dot{\epsilon} - \epsilon') \tag{2.8b}$$

$$\begin{aligned} \delta\pi = {} & \epsilon[(1 + 2\lambda)(\ddot{\phi} - \dot{\phi}') - 2\lambda(\dot{\phi}' - \phi'') \\ & + 2(\dot{\phi} - \phi')(\dot{\lambda} - \lambda')] - \epsilon'[\dot{\phi} - \phi'] \end{aligned} \tag{2.8c}$$

$$\delta p_\lambda = 0 \tag{2.8d}$$

where λ can be made equal to any given function of x and t, for an appropriate choice of the gauge parameter ϵ. The Hamiltonian formulation of the theory in various gauges has been studied in ref. 10. In the next section we study the BRST formulation of the theory following ref. 13.

3. THE BRST FORMULATION

In considering the BRST formulation[13] of the Siegel action (2.1),[5,6,10] we first convert the total Hamiltonian density into the first-order Lagrangian density:

$$\begin{aligned}\mathcal{L}_{\mathrm{IO}} &= \pi\dot{\phi} + p_\lambda\dot{\lambda} - \mathcal{H}_{\mathrm{T}} \\ &= \pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}\phi'^2 + \frac{\lambda}{1+2\lambda}(\pi - \phi')^2\end{aligned} \tag{3.1}$$

The first-order action of the theory $S_{\mathrm{IO}} = \int \mathcal{L}_{\mathrm{IO}}\, dt\, dx$ with $\mathcal{L}_{\mathrm{IO}}$ given by (3.1) is seen, with the use of $\pi = \partial\mathcal{L}/\partial\dot{\phi} = [\dot{\phi} + 2\lambda(\dot{\phi} - \phi')]$,[10] to be explicitly invariant under the gauge transformations (2.8).

3.1. Chiral Bosons and BRST Invariance

Following ref. 13, we rewrite our gauge-invariant theory[5,10] as a quantum system which possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge-invariant model[5,10] and replace the notion of a gauge transformation which shifts operators by c-number functions by a BRST transformation which mixes operators having different statistics. We then introduce new anticommuting variables c and $\bar{c}$ called Faddeev–Popov ghost and antighost fields, respectively (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b called the Nakanishi–Lautrup field such that

$$\hat{\delta}\phi = c(\dot{\phi} - \phi') \tag{3.2a}$$

$$\hat{\delta}\lambda = -\tfrac{1}{2}(\dot{c} + c') + c(\dot{\lambda} - \lambda') - \lambda(\dot{c} - c') \tag{3.2b}$$

$$\begin{aligned}\hat{\delta}\pi = {}& c[(1+2\lambda)(\ddot{\phi} - \dot{\phi}') - 2\lambda(\dot{\phi}' - \phi'') \\ &+ 2(\dot{\phi} - \phi')(\dot{\lambda} - \lambda')] - c'[\dot{\phi} - \phi']\end{aligned} \tag{3.2c}$$

$$\hat{\delta}p_\lambda = 0, \qquad \hat{\delta}c = c(\dot{c} - c'), \qquad \hat{\delta}\bar{c} = b, \qquad \hat{\delta}b = 0 \tag{3.2d}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function $f(\pi, p_\lambda, P_b, \pi_c, \pi_{\bar{c}}, \phi, \lambda, b, c, \bar{c})$ such that $\hat{\delta} f = 0$.

3.2. Gauge Fixing in the BRST Formalism

Performing gauge fixing in the BRST formalism implies adding to the first-order Lagrangian density (3.1) a trivial BRST-invariant function.[13] We thus write the quantum Lagrangian density

$$
\begin{aligned}
\mathscr{L}_{\mathrm{BRST}} &= \mathscr{L}_{\mathrm{IO}} + \hat{\delta}\left[\bar{c}\left(-2\dot{\lambda} + \frac{1}{2}b\right)\right] \\
&= \pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}\phi'^2 + \frac{\lambda}{1+2\lambda}(\pi - \phi')^2 \\
&\quad + \hat{\delta}\left[\bar{c}\left(-2\dot{\lambda} + \frac{1}{2}b\right)\right]
\end{aligned}
\tag{3.3}
$$

The last term in Eq. (3.3) is the extra BRST-invariant gauge-fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathscr{L}_{\mathrm{BRST}}$ (with some partial integrations):

$$
\begin{aligned}
\mathscr{L}_{\mathrm{BRST}} &= \pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}\phi'^2 + \frac{\lambda}{1+2\lambda}(\pi - \phi')^2 \\
&\quad + \frac{1}{2}b^2 - 2b\dot{\lambda} + \dot{\bar{c}}(\dot{c} + c')
\end{aligned}
\tag{3.4}
$$

The quantum action $S_{\mathrm{BRST}} = \int \mathscr{L}_{\mathrm{BRST}}\, dt\, dx$ with $\mathscr{L}_{\mathrm{BRST}}$ given by (3.3) or (3.4) is explicitly invariant under the BRST transformations (3.2). Proceeding classically, the Euler–Lagrange equation for b reads

$$
b = 2\dot{\lambda}
\tag{3.5}
$$

Also, the requirement $\hat{\delta} b = 0$ [cf. Eq. (3.2d)] implies

$$
\hat{\delta} b = 2\hat{\delta}\dot{\lambda} = 0
\tag{3.6}
$$

which in turn implies

$$
\ddot{c} + \dot{c}' = 0
\tag{3.7}
$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathscr{L}_{\mathrm{BRST}}$ with respect to $\bar{c}$. In introducing momenta we have to be careful in defining those for fermionic variables. Thus we define the bosonic momenta in the usual way so that

$$p_\lambda = \frac{\partial}{\partial \dot{\lambda}} \mathcal{L}_{\text{BRST}} = -2b \tag{3.8}$$

but for the fermionic momenta with directional derivatives we set

$$\pi_c := \mathcal{L}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \dot{c}} = \dot{\bar{c}}; \qquad \pi_{\bar{c}} := \frac{\overline{\partial}}{\partial \dot{\bar{c}}} \mathcal{L}_{\text{BRST}} = (\dot{c} + c') \tag{3.9}$$

implying that the variable canonically conjugate to c is $\dot{\bar{c}}$ and the variable conjugate to $\bar{c}$ is $(\dot{c} + c')$. In forming the Hamiltonian density $\mathcal{H}_c$ from the Lagrangian density in the usual way we remember that the former has to be Hermitian. Then

$$\begin{aligned} \mathcal{H}_{\text{BRST}} &= \pi\dot{\phi} + p_\lambda\dot{\lambda} + \pi_c\dot{c} + \dot{\bar{c}}\pi_{\bar{c}} - \mathcal{L}_{\text{BRST}} \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2 - \frac{\lambda}{1 + 2\lambda}(\pi - \phi')^2 - \frac{1}{8}p_\lambda^2 + \pi_c(\pi_{\bar{c}} - c') \end{aligned} \tag{3.10}$$

We can check the consistency of (3.9) with (3.10) by looking at Hamilton's equations for the fermionic variables, i.e.,

$$\dot{c} = \frac{\overline{\partial}}{\partial \pi_c} \mathcal{H}_{\text{BRST}}, \qquad \dot{\bar{c}} = \mathcal{H}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \pi_{\bar{c}}} \tag{3.11}$$

Thus

$$\dot{c} = \frac{\overline{\partial}}{\partial \pi_c} \mathcal{H}_{\text{BRST}} = (\pi_{\bar{c}} - c'); \qquad \dot{\bar{c}} = \mathcal{H}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \pi_{\bar{c}}} = \pi_c \tag{3.12}$$

in agreement with (3.9). The fermionic variables are assumed to anticommute so that

$$\{\pi_c, \pi_{\bar{c}}\} = \{\bar{c}, c\} = 0; \qquad \frac{d}{dt}\{\bar{c}, c\} = 0 \quad \text{or} \quad \{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\} \tag{3.13}$$

where { , } means anticommutator. Demanding that c satisfy the Heisenberg equation

$$[c, \mathcal{H}_{\text{BRST}}] = i\dot{c} \tag{3.14}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathcal{H}_{\text{BRST}}] = \{\dot{\bar{c}}, c\}\,\dot{c} \tag{3.15}$$

Then Eqs. (3.13)–(3.15) imply

$$\{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\} = i \tag{3.16}$$

Here the minus sign in the above equation implies the existence of states with negative norm (in the fermionic sector) in the space of state vectors of the theory.[13] The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{\mathrm{BRST}}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

3.3 The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformations (3.2). It mixes operators which satisfy Bose and Fermi statistics and its commutators with Bose operators, and its anticommutators with Fermi operators in the present case satisfy[13]

$$[\phi, Q] = \left[\frac{-2c(\pi - \phi')}{(1 + 2\lambda)^2}\right]; \qquad [p_\lambda, Q] = \left[\frac{-4c(\pi - \phi')^2}{(1 + 2\lambda)^3}\right] \tag{3.17a}$$

$$[\pi, Q] = \left[\frac{2c(\pi' - \phi'')}{(1 + 2\lambda)^2} - \frac{8c\lambda'(\pi - \phi')}{(1 + 2\lambda)^3} + \frac{2c'(\pi - \phi')}{(1 + 2\lambda)^2}\right] \tag{3.17b}$$

$$[\lambda, Q] = \dot{c}; \qquad \{\dot{\bar{c}}, Q\} = -\left[\frac{(\pi - \phi')^2}{(1 + 2\lambda)^2}\right] \tag{3.17c}$$

where all other commutators and anticommutators involving Q vanish. In view of (3.17), following ref. 13, the BRST charge operator of the present theory can be written as

$$Q = \int dx \left\{ ic\left[\frac{(\pi - \phi')^2}{(1 + 2\lambda)^2}\right] - i\dot{c}p_\lambda \right\} \tag{3.18}$$

It is easy to see that Q is nilpotent and therefore satisfies $Q^2 = 0$ and that it also commutes with H_{BRST}. The nilpotency of Q and $\bar{Q}$ [defined later by (3.20)] follows as a consequence of the fact that c, $\bar{c}$, $\dot{c}$, and $\dot{\bar{c}}$ are Grassmann variables so that $c^2 = \bar{c}^2 = \dot{c}^2 = \dot{\bar{c}}^2 = 0$, implying that $\{\bar{c}, \dot{c}\} = \{\dot{\bar{c}}, c\} = 0$, which, in turn, implies that $2Q^2 = \{Q, Q\} = 2\bar{Q}^2 = \{\bar{Q}, \bar{Q}\} = 0$. Further, Eq. (3.18) implies that the set of states satisfying the conditions $\Omega_1|\psi\rangle = 0$ and $\tilde{\Omega}_2|\psi\rangle = 0$ (or $\Omega_2|\psi\rangle = 0$) belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST-invariant states. Also, because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which $\Omega_1|\psi\rangle = 0$ and $\tilde{\Omega}_2|\psi\rangle = 0$ (or $\Omega_2|\psi\rangle = 0$), but also additional states for which $c|\psi\rangle = \dot{c}|\psi\rangle = 0$, with $\Omega_1|\psi\rangle \neq 0$

and $\tilde{\Omega}_2|\psi\rangle \neq 0$ (or $\Omega_2|\psi\rangle \neq 0$). However, the Hamiltonian is also invariant under the anti-BRST transformations (in which the role of c and $-\bar{c}$ is interchanged) given by

$$\bar{\hat{\delta}}\phi = -\bar{c}(\phi - \phi') \tag{3.19a}$$

$$\bar{\hat{\delta}}\lambda = \tfrac{1}{2}(\dot{\bar{c}} + \bar{c}') - \bar{c}(\dot{\lambda} - \lambda') + \lambda(\dot{\bar{c}} + \bar{c}') \tag{3.19b}$$

$$\begin{aligned}\bar{\hat{\delta}}\pi = & -\bar{c}[(1 + 2\lambda)(\ddot{\phi} - \dot{\phi}') - 2\lambda(\dot{\phi}' - \phi'') \\ & + 2(\dot{\phi} - \phi')(\dot{\lambda} - \lambda')] + \bar{c}'[\dot{\phi} - \phi']\end{aligned} \tag{3.19c}$$

$$\bar{\hat{\delta}}p_\lambda = 0, \qquad \bar{\hat{\delta}}\bar{c} = -\bar{c}(\dot{\bar{c}} - \bar{c}'), \qquad \bar{\hat{\delta}}c = -b, \qquad \bar{\hat{\delta}}b = 0 \tag{3.19d}$$

with generator or anti-BRST charge

$$\bar{Q} = \int dx \left\{ -i\bar{c}\left[\frac{(\pi - \phi')^2}{(1 + 2\lambda)^2}\right] + i\dot{\bar{c}}p_\lambda \right\} \tag{3.20}$$

We now have $[Q, H] = 0$, as well as $[\bar{Q}, H] = 0$, and we impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \qquad \text{and} \qquad \bar{Q}|\psi\rangle = 0 \tag{3.21}$$

The states for which $\Omega_1|\psi\rangle = 0$ and $\tilde{\Omega}_2|\psi\rangle = 0$ (or $\Omega_2|\psi\rangle = 0$) satisfy both of these conditions and, in fact, are the *only* states satisfying both of the above conditions because in view of Eq. (3.16), we cannot have simultaneously c, $\dot{c}$ and $\bar{c}$, $\dot{\bar{c}}$ applied to $|\psi\rangle$ to give zero. Thus we see that the only states satisfying (3.21) are those that satisfy the constraints of the theory $\Omega_1 = 0$ and $\tilde{\Omega}_2 = 0$ (or $\Omega_2 = 0$), and also that these states belong to the set of BRST-invariant and anti-BRST-invariant states.

It is important to observe here that when we study the usual Hamiltonian formulation of a gauge-invariant theory (like the present one) under some gauge-fixing conditions, we necessarily destroy the gauge invariance of the theory. However, in the BRST formulation when we imbed the gauge-invarant theory ($\mathcal{L}$) into a BRST-invariant system, the Hamiltonian density $\mathcal{H}_{\text{BRST}}$ (which includes the gauge-fixing contribution) commutes with Q as well as with $\bar{Q}$. The new symmetry which replaces the gauge invariance is maintained and hence projecting any state onto the sector of BRST- and anti-BRST-invariant states yields a theory which is isomorphic to $\mathcal{L}$.

The BRST quantization of the original Siegel action defined by (2.1),[5] which is the Siegel action without any modifications, is thus complete.

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